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A note on parasupersymmetric quantum mechanics of arbitrary order

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Abstract
We revisit parasupersymmetric quantum mechanics of arbitrary order and present a set of nontrivial relations, which characterizes the most general multilinear part of the associated parasupersymmetric algebra. We then show that the formulation of multilinear relations leads immediately to a polynomial of parasupersymmetric Hamiltonian in terms of the corresponding parasupercharges. The deduction of higher derivative supersymmetric quantum mechanics directly via this parasupersymmetric formulation is discussed. The complete degenerate structure of the energy spectrum for parasupersymmetric quantum mechanics of order p is systematically analyzed. Finally, the notion of cyclic symmetry is introduced and the algebra of cyclic charge operators of arbitrary order is developed, based on the parasupersymmetric formalism.

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1. Introduction

Supersymmetry is the symmetry between bosonic and fermionic degrees of freedom. The idea of supersymmetry was initially introduced to solve the hierarchy problem in quantum field theories and has been playing a significant role in many branches of physics since that time. In quantum mechanical systems, supersymmetric quantum mechanics (SQM) was investigated as a testing ground to understand non-perturbatively supersymmetry-breaking [1]. In particular, the technique of SQM allows us to establish various important properties in mathematical physics, such as the class of solvable potentials, the degeneracy of the energy spectrum, the relations among isospectral Hamiltonians, etc [2–6]. For a review of SQM, please refer to [7–9] and references therein.

The basic property of one-dimensional SQM of one boson and one fermion degrees of freedom is easily motivated by the fermionic annihilation and creation operators f and $f^\dagger$, which satisfy the commutation relations

$$(f)^2 = 0 = (f^\dagger)^2, \qquad \{f, f^\dagger\} = ff^\dagger + f^\dagger f = 1, \tag{1}$$

and are conveniently represented by the 2×2 matrices:

$$f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad f^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{2}$$

In terms of f and $f^\dagger$, we readily formulate two supercharges Q and $Q^\dagger$ of SQM by

$$Q = Af = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, \qquad Q^\dagger = A^\dagger f^\dagger = \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix}, \tag{3}$$

where $A = \hat{p} + \mathrm{i}W(x)$ and $A^\dagger = \hat{p} - \mathrm{i}W(x)$ are the first-order differential operators. Here, $\hat{p} = -\mathrm{i}\partial_x$ is the momentum operator and $W(x)$ is the superpotential.

Based on the two supercharges Q and $Q^\dagger$ (3), the algebra of one-dimensional SQM can be described as

$$Q^2 = 0 = (Q^\dagger)^2, \qquad [H, Q] = 0 = [H, Q^\dagger], \qquad QQ^\dagger + Q^\dagger Q = 2H, \tag{4}$$

where H is the supersymmetric Hamiltonian and takes the form

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}A^\dagger A & 0 \\ 0 & \frac{1}{2}AA^\dagger \end{pmatrix}. \tag{5}$$

It is known that supersymmetry may or may not be broken. The situation is characterized by the Witten index Δ_F, which is defined by the difference between the number of zero-energy states of the partner Hamiltonians H_1 and H_2. The same Witten index can also be defined by the asymptotic behavior of the superpotential $W(x)$ [8]. In a supersymmetric nonperiodic quantum system in one dimension, the consequence of equation (4) is that, for unbroken supersymmetry ($\Delta_F \neq 0$), all the positive energy states are twofold degenerate and the zero-energy ground state is always nondegenerate. In addition, for broken supersymmetry ($\Delta_F = 0$), there is no zero-energy ground state and all energy eigenstates are therefore twofold degenerate. However, it was pointed out in [10, 11] that a supersymmetric periodic quantum system may produce two zero-energy ground states, resulting in a completely isospectral pair of partner Hamiltonians. If this happens, we will have $\Delta_F = 0$ even in the case of unbroken supersymmetry. A typical example of supersymmetric periodic quantum systems investigated is that the partner potentials have identical band structures, and are self-isospectral in the sense that the potentials are the same in shape but differ merely by a half-period translation. Other self-isospectral examples can be found in the recent articles [12].

We note that in one-dimensional SQM the statement that all positive energy levels are twofold degenerate is valid only in the case of a discrete spectrum. If the spectra of the partner Hamiltonians admit a continuous spectrum, the corresponding energy levels will become fourfold degenerate. Generalized from this property, it can be shown that supersymmetric periodic quantum systems with a parity-even finite-gap potential can exhibit a very peculiar, the so-called tri-supersymmetric structure that originates from a hidden bosonized supersymmetry [12, 13].

Besides the ordinary supersymmetry, there is an extended version called parasupersymmetry [14], based on the notion of parastatistics [15]. In brevity, parasupersymmetry of order p describes the symmetry between bosons and parafermions of order p, and admits intrinsically p copies of ordinary supersymmetry. The algebra of parasupersymmetric quantum mechanics (PSQM) of order 2 was introduced by Rubakov and Spiridonov [14]. The formulation of PSQM of arbitrary order has been generalized by Khare [16], and with a modified version by Beckers and Debergh [17]. Many important results have since been obtained [18–23]. Though it is a direct generalization of the ordinary SQM, the PSQM actually suffers from one unsatisfactory feature that does not occur in the ordinary SQM. The unsatisfactory feature is that the parasupersymmetric Hamiltonian cannot be directly expressed in terms of the associated parasupercharges. As far as we are aware, this problem has remained unsolved until today.

The purpose of the present paper is to take a fresh look at this problem and to resolve it. We will first establish a general set of multilinear relations which the parafermionic creation and annihilation operators $a^\dagger$ and a fulfill. Then we show that a similar set of multilinear relations exists for the parasupercharges Q and $Q^\dagger$, too. One of the key features regarding the latter multilinear relations is that a polynomial combination of the parasupersymmetric Hamiltonian H is found expressible in terms of the corresponding parasupercharges Q and $Q^\dagger$. Based on our parasupersymmetric formulation, a brief discussion on the derivation of higher derivative SQM will be given, and the complete degenerate structure of the energy spectrum for PSQM of order p will be analyzed in a systematic way. At the end of the paper, we introduce the notion of cyclic charge operators of arbitrary order and develop its associated quantum mechanical algebra.

The paper is organized as follows. In section 2, we establish the general set of multilinear relations between the parasupersymmetric Hamiltonian and the associated parasupercharges. Some important aspects derivable from these multilinear relations are discussed. In section 3, the degenerate structure of the energy spectrum for PSQM is analyzed in detail. In section 4, the notion of cyclic symmetry of arbitrary order is introduced and the associated algebra of the cyclic operators is constructed. Section 5 is devoted to a discussion of the obtained results.

2. Relations between the parasupersymmetric Hamiltonian and parasupercharges

This section contains two parts. In the first part, to have a self-contained presentation, we briefly review and reproduce the relevant properties of PSQM of order p of [16]. Some of the results are quoted without proof for brevity. In the second part, we construct a general set of multilinear relations that are satisfied by the parasupersymmetric charges and the parasupersymmetric Hamiltonian. Two remarks concerning the multilinear relations will be made at the end of the section.

The parasupersymmetric generalization of the ordinary SQM algebra is straightforward. It is achieved by replacing the fermionic operators f and $f^\dagger$ in equation (1) by the corresponding parafermionic counterparts a and $a^\dagger$ of arbitrary order p $(p = 1, 2, 3, \ldots)$. The parafermionic creation and annihilation operators $a^\dagger$ and a are known to satisfy the algebra [24]

$$(a)^{p+1} = 0 = (a^\dagger)^{p+1}, \qquad [[a^\dagger, a], a] = -2a, \qquad [[a^\dagger, a], a^\dagger] = 2a^\dagger, \tag{6}$$

as well as the following nontrivial multilinear relation (and its Hermitian-conjugated one) [16]

$$a^p a^\dagger + a^{p-1}a^\dagger a + \cdots + a a^\dagger a^{p-1} + a^\dagger a^p = \tfrac{1}{6}p(p+1)(p+2)a^{p-1}, \tag{7}$$

where the left-hand side has $(p + 1)$ terms. When $p = 1$, it obviously reduces to equation (1), that is, the bilinear relation of the fermionic operators f and $f^\dagger$.

At this stage, a useful representation for the parafermionic operators a and $a^\dagger$, which fulfills equations (6) and (7), can be given by $(p + 1) \times (p + 1)$ matrices:

$$(a)_{\alpha\beta} = C_\beta \delta_{\alpha,\beta+1}, \qquad (a^\dagger)_{\alpha\beta} = C_{\beta-1}\delta_{\alpha+1,\beta}, \tag{8}$$

where $\alpha, \beta = 1, 2, \ldots, (p + 1)$ and the coefficients C_β are

$$C_\beta = \sqrt{\beta(p - \beta + 1)} = C_{p-\beta+1}. \tag{9}$$

Utilizing the above matrix realization (8), we formulate two parasupersymmetric charges Q and $Q^\dagger$ of PSQM of order p, much in the same way as the case in the SQM algebra. They are, respectively,

$$(Q)_{\alpha\beta} = A_\beta \delta_{\alpha,\beta+1}, \qquad (Q^\dagger)_{\alpha\beta} = A^\dagger_{\beta-1}\delta_{\alpha+1,\beta}, \tag{10}$$

where $A_\beta = \hat{p} - \mathrm{i}W_\beta(x)$ and $A_\beta^\dagger = \hat{p} + \mathrm{i}W_\beta(x)$ are p pairs of first-order differential operators, for $\beta = 1$ to p. The properties of the p superpotentials $W_\beta(x)$ are to be specified.

Then with the help of the matrix forms for Q and $Q^\dagger$ (10), one readily finds that the algebra of PSQM of one boson and one parafermion of order p is characterized by the relations

$$(Q)^{p+1} = 0 = (Q^\dagger)^{p+1}, \qquad [H, Q] = 0 = [H, Q^\dagger], \tag{11}$$

$$Q^p Q^\dagger + Q^{p-1} Q^\dagger Q + \cdots + Q Q^\dagger Q^{p-1} + Q^\dagger Q^p = 2pHQ^{p-1}, \tag{12}$$

and the Hermitian-conjugated one of equation (12). Here, the parasupersymmetric Hamiltonian H is a diagonal $(p+1) \times (p+1)$ matrix:

$$(H)_{\alpha\beta} = H_\alpha \delta_{\alpha\beta}, \tag{13}$$

with the diagonal elements given by $(i = 1, 2, \ldots, p)$

$$H_i = \tfrac{1}{2} A_i^\dagger A_i + c_i \qquad \text{and} \qquad H_{p+1} = \tfrac{1}{2} A_p A_p^\dagger + c_p, \tag{14}$$

where $c_1, c_2, \ldots, c_p$ are arbitrary constants that to be consistent with equation (12) are related to one another by the identity $\sum_{i=1}^{p} c_i = 0$. Meanwhile, the requirement that the parasupersymmetric Hamiltonian H commute with the parasupercharges Q and $Q^\dagger$ imposes $(p-1)$ conditions on the p superpotentials $(k = 2, 3, \ldots, p)$:

$$\tfrac{1}{2} A_{k-1} A_{k-1}^\dagger + c_{k-1} = \tfrac{1}{2} A_k^\dagger A_k + c_k. \tag{15}$$

Let us summarize here some basic properties of PSQM of order p. Equations (11)–(15) are sufficient to prove the following statements. (i) The spectrum of the parasupersymmetric Hamiltonian H of order p is $(p+1)$-fold degenerate at least starting from the pth and higher excited states. (ii) The nature of the ground state and the first $(p-1)$ excited states depends on the specific form of the p superpotentials $W_\beta(x)$. (iii) There are p ordinary super-Hamiltonians that can be associated with the parasupersymmetric Hamiltonian of order p. (iv) For some particular superpotentials, the parasupersymmetric Hamiltonian of order p describes one-dimensional motion of a spin-$\frac{p}{2}$ particle in a magnetic field. (v) Apart from the parasupercharges Q and $Q^\dagger$ defined in equation (10), there exist $(p-1)$ other sets of conserved, independent parasupercharges, all of which also satisfy the PSQM algebra (11) and (12).

As mentioned earlier, the formulation of PSQM described has an unsatisfactory feature, which is that, unlike in SQM, we are unable to write the parasupersymmetric Hamiltonian H (13) directly in terms of the associated parasupercharges Q and $Q^\dagger$. Because the inverse of Q^{p-1} dose not exist, equation (12) cannot be simply inverted to get the parasupersymmetric Hamiltonian H on one side and the parasupercharges Q and $Q^\dagger$ on the other side.

To solve this problem, let us choose to consider a more general set of multilinear relations satisfied by the parafermionic operators a and $a^\dagger$ of order p. The general set of relations, that by construction contains equation (7) as a special case, is given by

$$\sum_{m=0}^{p} a^{p-m} a^{\dagger\, p-q} a^m = a^p a^{\dagger\, p-q} + a^{p-1} a^{\dagger\, p-q} a + \cdots + a a^{\dagger\, p-q} a^{p-1} + a^{\dagger\, p-q} a^p, \tag{16}$$

where q is an integer and takes values in the interval $[\,0, p-1\,]$. There are $(p+1)$ terms on the right-hand side. Obviously, equation (16) equals the left-hand side of equation (7) for $q = p-1$ and becomes trivial identity for either $q = p$ or $q \leqslant -1$.

Now making use of equation (8) for the parafermionic operators a and $a^\dagger$, we easily obtain in matrix components the set of relations defined by equation (16) as[1]

$$\sum_{m=0}^{p}(a^{p-m}a^{\dagger\, p-q}a^m)_{\alpha\beta} = \sum_{m=0}^{p}\sum_{\sigma,\tau}\left[\prod_{i=1}^{p-m}C_{\sigma+i-1}\right]\delta_{\alpha,\sigma+p-m} \times\left[\prod_{j=1}^{p-q}C_{\sigma+j-1}\right]\delta_{\sigma+p-q,\tau}\left[\prod_{k=1}^{m}C_{\beta+k-1}\right]\delta_{\tau,\beta+m}, \tag{17}$$

where $\alpha, \beta = 1, 2, \ldots, (p+1)$. Because of the presence of Kronecker deltas, the summation in the index τ can be performed immediately. The outcome is that the summation indices m and σ are constrained within the region of parallelogram: $\max(1, q+m+1-p) \leqslant \sigma \leqslant \min(m+1, q+1)$. At the same time, the index α takes values in the interval $[\,q+1, p+1\,]$ and the index $\beta = \alpha - q$. The summation over m can then be carried out, resulting in the index σ in the allowed range: $1 \leqslant \sigma \leqslant q+1$, for a fixed α. After performing both summations, we arrive at

$$\begin{aligned}\sum_{m=0}^{p}(a^{p-m}a^{\dagger\, p-q}a^m)_{\alpha\beta} &= \sum_{\sigma=1}^{q+1}\left[\prod_{i=\sigma}^{\alpha-1}C_i\right]\left[\prod_{j=\sigma}^{\sigma+p-q-1}C_j\right]\left[\prod_{k=\alpha-q}^{\sigma+p-q-1}C_k\right]\delta_{\alpha,\beta+q},\\ &= \sum_{\sigma=1}^{q+1}\left[\prod_{i=\sigma}^{\sigma+p-q-1}C_i^2\right](a^q)_{\alpha\beta},\end{aligned} \tag{18}$$

where we have used equation (8) and the identity $\prod_{k=\alpha-q}^{\alpha-1}C_k = \prod_{k=1}^{q}C_{\beta+k-1}$.

An important observation regarding equation (18) is that the set of multilinear relations (16) as a whole behaves like a definite $(p+1)\times(p+1)$ matrix. It is a matrix proportional to the qth powers of the parafermionic annihilation operator a, with the proportionality constant

$$\sum_{\sigma=1}^{q+1}\left[\prod_{i=\sigma}^{\sigma+p-q-1}C_i^2\right] = ((p-q)!)^2\binom{2p-q+1}{q}. \tag{19}$$

Here, equation (9) has been used for the computation of the coefficient C_i^2 and the symbol $\binom{m}{n} = m!/n!(m-n)!$ denotes the binomial coefficient. Consequently, by combining equations (18) and (19), we establish the needed result of the multilinear relations (plus the Hermitian-conjugated ones) of the parafermionic operators a and $a^\dagger$ as

$$\sum_{m=0}^{p}a^{p-m}a^{\dagger\, p-q}a^m = ((p-q)!)^2\binom{2p-q+1}{q}a^q. \tag{20}$$

It is easily checked that when $q = p-1$, equation (20) as desired reproduces equation (7). Especially, when $q = 0$ the right-hand side of equation (20) contains no parafermionic operator a and becomes a pure number! This special relation reads

$$\sum_{m=0}^{p}a^{p-m}a^{\dagger\, p}a^m = (p!)^2. \tag{21}$$

[1] From the matrix representation (8), the mth powers of the parafermionic operators a and $a^\dagger$ are, respectively,

$$(a^m)_{\alpha\beta} = \left[\prod_{i=1}^{m}C_{\beta+i-1}\right]\delta_{\alpha,\beta+m}, \qquad (a^{\dagger\, m})_{\alpha\beta} = \left[\prod_{i=1}^{m}C_{\beta-i}\right]\delta_{\alpha+m,\beta}.$$

The existence of the set of multilinear relations for the parafermionic operators a and $a^\dagger$ (20) strongly suggests that we may have a similar set of multilinear relations for the parasupercharges Q and $Q^\dagger$ in the algebra of PSQM of order p. Further, such a set if it exists in the PSQM is expected to be very compact, as well. Hence, let us try to construct the set of multilinear relations for the parasupercharges. Similar to equation (16), we write the general set of multilinear relations satisfied by the parasupercharges Q and $Q^\dagger$ of order p by

$$\sum_{m=0}^{p} Q^{p-m} Q^{\dagger\, p-q} Q^m, \tag{22}$$

where q is an integer and takes values in the interval $[0, p-1]$. Note that equation (22) has $(p+1)$ terms and becomes equation (12) for $q = p-1$.

Equation (22) can be readily worked out by the matrix representation of parasupercharges Q and $Q^\dagger$ in equation (10). The computation is straightforward; hence, let us list the results for the first three values of order p. The case of order $p = 1$ is trivial, and is shown in equation (4). The first nontrivial case is of order $p = 2$. After a short algebra, we find

$$Q^2 Q^\dagger + Q Q^\dagger Q + Q^\dagger Q^2 = 2\sum_{i=1}^{2}(H - c_i)Q, \tag{23}$$

$$Q^2 Q^{\dagger 2} + Q Q^{\dagger 2} Q + Q^{\dagger 2} Q^2 = 2^2 \prod_{i=1}^{2}(H - c_i), \tag{24}$$

where $c_1 + c_2 = 0$ is understood. The next nontrivial case is $p = 3$. It is not hard to construct these relations (with $c_1 + c_2 + c_3 = 0$)

$$Q^3 Q^\dagger + Q^2 Q^\dagger Q + Q Q^\dagger Q^2 + Q^\dagger Q^3 = 2\sum_{i=1}^{3}(H - c_i)Q^2, \tag{25}$$

$$Q^3 Q^{\dagger 2} + Q^2 Q^{\dagger 2} Q + Q Q^{\dagger 2} Q^2 + Q^{\dagger 2} Q^3 = 2^2 \sum_{i=1}^{2}\left[\prod_{j=i}^{i+1}(H - c_j)\right]Q, \tag{26}$$

$$Q^3 Q^{\dagger 3} + Q^2 Q^{\dagger 3} Q + Q Q^{\dagger 3} Q^2 + Q^{\dagger 3} Q^3 = 2^3 \prod_{i=1}^{3}(H - c_i). \tag{27}$$

The computation of equation (22) extended to arbitrary order p can be similarly calculated. At the end of the day, we establish the general set of multilinear relations fulfilled by the parasupercharges Q and $Q^\dagger$ of arbitrary order p (and its Hermitian-conjugated set) in this compact expression

$$\sum_{m=0}^{p} Q^{p-m} Q^{\dagger\, p-q} Q^m = 2^{p-q} \sum_{i=1}^{q+1}\left[\prod_{j=i}^{i+p-1-q}(H - c_j)\right]Q^q, \tag{28}$$

where H is the parasupersymmetric Hamiltonian defined in equation (13) and c_i are the same constants given in equation (14), satisfying $\sum_{i=1}^{p} c_i = 0$.

It is now easily checked from equation (28) that when $q = p-1$, we recover equation (12). In particular, when $q = 0$ we obtain an instructive relation that is a homogeneous bi-multilinear relation in Q and $Q^\dagger$ as

$$Q^p Q^{\dagger p} + Q^{p-1} Q^{\dagger p} Q + \cdots + Q Q^{\dagger p} Q^{p-1} + Q^{\dagger p} Q^p = 2^p \prod_{i=1}^{p}(H - c_i). \tag{29}$$

Note that the right-hand side of equation (29) contains solely the parasupersymmetric Hamiltonian H, and no parasupercharges Q or $Q^\dagger$. In other words, via this equation a polynomial form of the parasupersymmetric Hamiltonian H of order p is proved expressible in terms of the corresponding parasupercharges Q and $Q^\dagger$.

Two remarks concerning equation (29) are in order at this stage. Other important aspects of equation (29) are presented in the following two sections.

(1) It seems that equation (29) is more suitable than equation (12) to represent the generalized version of the anticommutation relation, that is, $QQ^\dagger + Q^\dagger Q = 2H$, in the SQM algebra. The reason is that equation (29) and the anticommutation relation in SQM share the same equational structure, where the supercharges (parasupercharges) are on one side and the super-Hamiltonian (parasupersymmetric Hamiltonian) is on the other side of the equation. Therefore, we can restate the algebra of PSQM of order p by using equation (29) as follows. The algebra of PSQM of order p is characterized by the parasupercharges Q and $Q^\dagger$, which satisfy equation (11) as well as the nontrivial multilinear relation (29).

(2) Based on equation (29), we can readily translate the formulation of PSQM of order p into that of the higher derivative supersymmetric quantum mechanics (HSQM) of order p. Here, the term 'higher derivative SQM' is also named as 'nonlinear SQM' in the literature of supersymmetry. Since the relation between parasupersymmetry and nonlinear supersymmetry has been reported [25, 26], we thus concentrate on how equation (29) really works. Technically, HSQM is characterized by the standard supersymmetry algebra, but with supercharges involve higher order differential operators [27]. In the literature [28–31], the second-order HSQM has been discussed extensively in the context of spectral design, where one or two more energy levels are allowed to create above the ground-state energy level of the initial Hamiltonian. It is also possible to generate supersymmetric complex potentials with real energy eigenvalues in the formalism of HSQM. Moreover, it is recently found that a special nonlinear supersymmetry of the reflectionless Pöschl–Teller system is originated by the Aharonov–Bohm effect for a nonrelativistic particle on the AdS_2 surface. Consequently, the correspondence between these two quantum systems can be further explained within the framework of AdS/CFT holography [32].

The procedure of translation from PSQM to HSQM is as follows. We note that equation (29) represents a $(p+1)\times(p+1)$ matrix equation, which in components only has nonvanishing diagonal elements. To be more specific, the right-hand side of this equation is diagonal, since it is written purely in the parasupersymmetric Hamiltonian H. The left-hand side can also be shown to be diagonal, with the first diagonal element being $Q^{\dagger p}Q^p$, the second element $QQ^{\dagger p}Q^{p-1}$, the third element $Q^2Q^{\dagger p}Q^{p-2}, \ldots$, and the $(p+1)$st element $Q^pQ^{\dagger p}$. Now, the pth-order HSQM can be constructed with ease through equation (29) by truncating all of its intermediate components, and keeping only its first and last diagonal components alive. As a result, the original $(p+1)\times(p+1)$ matrix equation is reduced to a 2×2 one, which takes the simple form [26]

$$\{\tilde{Q}, \tilde{Q}^\dagger\} = \tilde{Q}\tilde{Q}^\dagger + \tilde{Q}^\dagger\tilde{Q} = 2^p \prod_{i=1}^{p}(\tilde{H} - c_i). \tag{30}$$

Here, $\tilde{H}$ as a 2×2 matrix defines the super-Hamiltonian of pth-order HSQM:

$$\tilde{H} = \begin{pmatrix} H_1 & 0 \\ 0 & H_{p+1} \end{pmatrix}, \tag{31}$$

and $\tilde{Q}$ and $\tilde{Q}^\dagger$ are pth-order supercharges constructed from the parasupercharges Q and $Q^\dagger$ via the identification $\tilde{Q} \sim Q^p$ and $\tilde{Q}^\dagger \sim Q^{\dagger p}$. Explicitly, they are given by the respective 2×2 matrices:

$$\tilde{Q} = \begin{pmatrix} 0 & 0 \\ A_p A_{p-1} \cdots A_1 & 0 \end{pmatrix}, \qquad \tilde{Q}^\dagger = \begin{pmatrix} 0 & A_1^\dagger A_2^\dagger \cdots A_p^\dagger \\ 0 & 0 \end{pmatrix}, \tag{32}$$

where both supercharges $\tilde{Q}$ and $\tilde{Q}^\dagger$ involve pth-order differential operators. The anticommutation relation (30) together with the relations $\tilde{Q}^2 = 0 = (\tilde{Q}^\dagger)^2$ and $[\tilde{Q}, \tilde{H}] = 0 = [\tilde{Q}^\dagger, \tilde{H}]$ defines the algebra of pth-order HSQM.

3. Degenerate structure of the spectrum

The degenerate structure of the energy spectrum of PSQM of order p is systematically discussed in this section. We will restrict our discussion on the parasupersymmetric quantum systems that admit only discrete spectrum, and no continuous one. It has been mentioned in section 2 that the spectrum of the parasupersymmetric Hamiltonian H of order p is $(p+1)$-fold degenerate at least starting from the pth and higher excited states, and that the degeneracy of the ground state and the first $(p-1)$ excited states depends on the specific form of the p superpotentials $W_i(x)$.

We hence divide the entire energy spectrum into an upper portion spectrum that is $(p+1)$-fold degenerate, and a lower portion one that is at most p-fold degenerate. In other words, the lower portion spectrum can be nondegenerate, twofold degenerate, . . ., or up to the maximal p-fold degenerate. The detailed structure of the lower portion spectrum will depend on the normalizability of the $2p$ functions $\psi_\pm^{(i)}(x)$ constructed in terms of p superpotentials [14]

$$\psi_\pm^{(i)}(x) = \exp\left[\pm \int^x W_i(x')\, \mathrm{d}x'\right], \tag{33}$$

where $i = 1, 2, \ldots, p$. Below, we will analyze various properties of the lower portion spectrum of PSQM, which include the energy eigenvalues, the number of possible degenerate cases, the reflection symmetry of the spectrum and the status of broken or unbroken parasupersymmetry. To learn some of these properties appropriately in PSQM, let us start with the simplest case, that is, the SQM.

(1) The case of $p = 1$. If supersymmetry is unbroken, the lower portion spectrum of SQM contains a single nondegenerate zero-energy ground state. If supersymmetry is broken, there is no such lower portion spectrum, because all the energy levels are twofold degenerate and by definition belong to the upper portion one. We can understand this fact by introducing a combined state for the lower portion spectrum as

$$|\Psi(1)\rangle = |n_1, \bar{n}_1\rangle = \begin{pmatrix} n_1 \psi_+^{(1)} \\ \bar{n}_1 \psi_-^{(1)} \end{pmatrix}, \tag{34}$$

where the pair of constants $(n_1, \bar{n}_1)$ can take values on $(0, 0)$, $(1, 0)$ or $(0, 1)$, depending on the normalizability of $\psi_+^{(1)}$ and $\psi_-^{(1)}$. Note that nonzero n_1 and $\bar{n}_1$ cannot coexist.

For $(n_1, \bar{n}_1) \neq (0, 0)$, the super-Hamiltonian H acting on the combined state $|n_1, \bar{n}_1\rangle$ renders a zero-energy result: $H|n_1, \bar{n}_1\rangle = 0$. Similarly, when equation (29) with $p = 1$, that is, $QQ^\dagger + Q^\dagger Q = 2H$, is applied on the same state $|n_1, \bar{n}_1\rangle$, the right-hand side vanishes automatically. To get a consistent result, we must have on the left-hand side of the equation that $Q|n_1, \bar{n}_1\rangle = Q^\dagger|n_1, \bar{n}_1\rangle = 0$. This results in the well-known condition on the ground state of the super-Hamiltonian for supersymmetry to be unbroken.

There are two possible ground states in the lower portion spectrum of SQM, namely $|n_1, 0\rangle$ or $|0, \bar{n}_1\rangle$. These two ground states can be mapped to each other, $|n_1, 0\rangle \leftrightarrow |0, \bar{n}_1\rangle$, by the reflection symmetry $n_1 \leftrightarrow \bar{n}_1$, which originates from the interchange of superpotential $W(x) \leftrightarrow -W(x)$. In short, for unbroken supersymmetry, we have two possible structures for the lower portion spectrum. Upon imposing the reflection symmetry, we are left with only one possibility. Finally, as far as the broken supersymmetry is concerned, it is signified by the choice of the pair $(n_1, \bar{n}_1) = (0, 0)$.

(2) The case of $p = 2$. The detailed structure of the energy spectrum of PSQM of order 2 has been studied [14]. It is known that its lower portion spectrum can be non-degenerate or twofold degenerate. Similar to the case of SQM, we can describe the complete structure of the lower portion spectrum by the combined state

$$|\Psi(2)\rangle = |n_1, \bar{n}_1 + n_2, \bar{n}_2\rangle = \begin{pmatrix} n_1\psi_+^{(1)} \\ \bar{n}_1\psi_-^{(1)} + n_2\psi_+^{(2)} \\ \bar{n}_2\psi_-^{(2)} \end{pmatrix}, \tag{35}$$

and by those states that are obtainable from this combined state by the action of the associated parasupercharges: $Q|\Psi(2)\rangle$ and $Q^\dagger|\Psi(2)\rangle$. Explicitly, we have

$$Q|\Psi(2)\rangle = \begin{pmatrix} 0 \\ 0 \\ \bar{n}_1 A_2\psi_-^{(1)} \end{pmatrix}, \qquad Q^\dagger|\Psi(2)\rangle = \begin{pmatrix} n_2 A_1^\dagger\psi_+^{(2)} \\ 0 \\ 0 \end{pmatrix}. \tag{36}$$

Here, the both pairs of constants $(n_1, \bar{n}_1)$ and $(n_2, \bar{n}_2)$ take values on $(0, 0)$, $(1, 0)$ or $(0, 1)$. Similarly, nonzero n_i and $\bar{n}_i$ cannot coexist, for $i = 1, 2$.
To proceed the discussion, we exclude for the moment the possibility that both pairs can vanish simultaneously, that is, $(n_1, \bar{n}_1) = (n_2, \bar{n}_2) = (0, 0)$ is excluded. Then, it can be shown that the parasupersymmetric Hamiltonian H obeys these equations

$$H|n_1, \bar{n}_1 + n_2, \bar{n}_2\rangle = c_1|n_1, \bar{n}_1, 0\rangle + c_2|0, n_2, \bar{n}_2\rangle, \tag{37}$$

$$(H - c_1)(H - c_2)|n_1, \bar{n}_1 + n_2, \bar{n}_2\rangle = 0. \tag{38}$$

Hence, the combined state (35) is a generic composition of eigenstates of energy c_1 and c_2, respectively. When the combined state is acted on by equation (24) (and its Hermitian-conjugated equation), we obtain the following equations, describing how this combined state can be kept invariant under the combinations of parasupercharges (for $m = 0, 1$):

$$Q^2 Q^{\dagger m}|n_1, \bar{n}_1 + n_2, \bar{n}_2\rangle = 0, \qquad Q^{\dagger 2} Q^m|n_1, \bar{n}_1 + n_2, \bar{n}_2\rangle = 0. \tag{39}$$

The result of equation (39) is quite instructive, since it represents the necessary condition for the lower portion spectrum of PSQM to preserve parasupersymmetry of order 2. We therefore reach the conclusion that equation (39) is a naturally parasupersymmetric generalization of the corresponding condition on the ground state for unbroken supersymmetry, that is, $Q|n_1, \bar{n}_1\rangle = Q^\dagger|n_1, \bar{n}_1\rangle = 0$. Obviously, the case of broken parasupersymmetry is detected by the choice $(n_1, \bar{n}_1) = (n_2, \bar{n}_2) = (0, 0)$, where equation (39) is not fulfilled.

Because each pair of $(n_i, \bar{n}_i)$ has three choices, the total number of possible structures for the lower portion spectrum is $(3^2 - 1) = 8$, for parasupersymmetry to be unbroken. The reflection symmetry of the lower portion spectrum can be denoted by the interchanges: $n_1 \leftrightarrow \bar{n}_2$ and $\bar{n}_1 \leftrightarrow n_2$, or equivalently, denoted by $W_1(x) \leftrightarrow -W_2(x)$ and $c_1 \leftrightarrow c_2$. Under the action of this reflection, the eight possible structures transform as

$$|n_1, 0, \bar{n}_2\rangle \leftrightarrow |n_1, 0, \bar{n}_2\rangle, \qquad |0, \bar{n}_1 + n_2, 0\rangle \leftrightarrow |0, \bar{n}_1 + n_2, 0\rangle, \tag{40}$$

$$|n_1, 0, 0\rangle \leftrightarrow |0, 0, \bar{n}_2\rangle, \quad |0, \bar{n}_1, 0\rangle \leftrightarrow |0, n_2, 0\rangle, \quad |n_1, n_2, 0\rangle \leftrightarrow |0, \bar{n}_1, \bar{n}_2\rangle. \tag{41}$$

Thus, only five distinct structures exist for the lower portion spectrum of PSQM of order 2. Among them, the lower portion spectrum described by the state $|0, \bar{n}_1 + n_2, 0\rangle$ is simplified, because for nonvanishing $\bar{n}_1$ and n_2 we must have $c_1 = c_2$ and $W_1(x) = -W_2(x)$. In summary, for unbroken parasupersymmetry of order 2, there are eight different possibilities for the lower portion spectrum, and only five survive by the reflection symmetry.

(3) The case of arbitrary order p. The lower portion of the spectrum of PSQM of order p can be at most p-fold degenerate. The detailed degenerate structure of the lower portion spectrum can be represented by the combined state

$$|\Psi(p)\rangle = |n_1, \bar{n}_1 + n_2, \bar{n}_2 + n_3, \ldots, \bar{n}_p\rangle = \begin{pmatrix} n_1\psi_+^{(1)} \\ \bar{n}_1\psi_-^{(1)} + n_2\psi_+^{(2)} \\ \bar{n}_2\psi_-^{(2)} + n_3\psi_+^{(3)} \\ \cdots \\ \bar{n}_p\psi_-^{(p)} \end{pmatrix}, \tag{42}$$

plus the states that are constructed from this combined state by successive application of the parasupercharges: $Q^m|\Psi(p)\rangle$ and $Q^{\dagger^m}|\Psi(p)\rangle$, where $m = 1, 2, \ldots, p-1$. Similarly, the p pairs of $(n_i, \bar{n}_i)$, for $i = 1, 2, \ldots, p$, take values on (0, 0), (1, 0) or (0, 1). Again, nonzero n_i and $\bar{n}_i$ cannot coexist, for each value of i.

When the pth-order parasupersymmetric Hamiltonian is applied on the above combined state $|\Psi(p)\rangle$, we get

$$H|n_1, \bar{n}_1 + n_2, \bar{n}_2 + n_3, \ldots, \bar{n}_p\rangle = \sum_{i=1}^{p} c_i|0, \ldots, 0, n_i, \bar{n}_i, 0, \ldots, 0\rangle, \tag{43}$$

$$\prod_{i=1}^{p}(H - c_i)|n_1, \bar{n}_1 + n_2, \bar{n}_2 + n_3, \ldots, \bar{n}_p\rangle = 0. \tag{44}$$

The combined state (42) is thus composed of the eigenstates of energy $c_1, c_2, \ldots$, and c_p. Further, when this combined state is acted on by equation (29) and the Hermitian-conjugated one, we obtain the necessary condition for the lower portion spectrum of PSQM to preserve parasupersymmetry of order p. The condition reads $(m = 0, 1, \ldots, p-1)$

$$Q^p Q^{\dagger^m}|n_1, \bar{n}_1 + n_2, \bar{n}_2 + n_3, \ldots, \bar{n}_p\rangle = 0, \tag{45}$$

$$Q^{\dagger^p} Q^m|n_1, \bar{n}_1 + n_2, \bar{n}_2 + n_3, \ldots, \bar{n}_p\rangle = 0. \tag{46}$$

Note that the breaking of parasupersymmetry of order p is marked by the choice of all constants $(n_i, \bar{n}_i) = (0, 0)$, for $i = 1$ to p.

For unbroken parasupersymmetry, the reflection symmetry of the lower portion spectrum is obtained by the interchanges: $n_i \leftrightarrow \bar{n}_{p+1-i}$, for $i = 1, 2, \ldots, p$. The same reflection can also be represented by the interchanges: $W_i(x) \leftrightarrow -W_{p+1-i}(x)$ and $c_i \leftrightarrow c_{p+1-i}$. In sum, it can be shown that there are $(3^p - 1)$ different structures for the lower portion spectrum, all of which respect parasupersymmetry of order p. By the reflection symmetry, the independent number becomes $\frac{1}{2}(3^p + 3^{[p/2]} - 2)$, where $[p/2] = m$, for $p = 2m$ or $p = 2m + 1$.

An interesting situation would occur, if we set all the constants c_1 to c_p vanishing in equation (29). Then, all the nonzero energy eigenstates in the lower portion spectrum will collapse to the eigenstate of zero energy. As a result, all the eigenstates in the upper portion of the spectrum have only positive energies. If this happens, equation (29) leads immediately to the fact that the parasupersymmetric Hamiltonian H of PSQM is directly expressed in terms of the associated parasupercharges Q and $Q^\dagger$ by

$$H = \frac{1}{2}\left[\sum_{m=0}^{p} Q^{p-m} Q^{\dagger p} Q^m\right]^{1/p}. \tag{47}$$

Remember that the terms on the left-hand side of equation (29) are already diagonal and non-negative, thus can be taken the pth root without causing any trouble. Equation (47) gives a simpler formulation of PSQM of order p, in which all the energy eigenvalues are non-negative and all the excited states are always $(p + 1)$-fold degenerate. Moreover, the ground state has zero energy and is at most p-fold degenerate. Note that the existence of nonzero vacuum energy in the Hamiltonian (47) signifies the breaking of parasupersymmetry. Let us mention that an alternative formulation of PSQM, which also leads to an explicit expression for the parasupersymmetric Hamiltonian in terms of the parasupercharges, is discussed in [33].

4. Cyclic symmetry

We introduce the notion of cyclic symmetry and construct the associated cyclic algebra of arbitrary order, based on the parasupersymmetric formulation established in section 2. The cyclic symmetry of order p is characterized by the cyclic operators b and $b^\dagger$, which obey the unity relation as [34, 35]

$$(b)^{p+1} = 1 = (b^\dagger)^{p+1}, \tag{48}$$

plus a set of multilinear relations (and the Hermitian-conjugated one) similar to equation (20). See equation (51) for clarity. The existence of the multilinear relations for b and $b^\dagger$ further suggests that we should have the similar relations for the cyclic quantum charges $\mathcal{Q}$ and $\mathcal{Q}^\dagger$ in the algebra of cyclic quantum mechanics of arbitrary order.

In fact, equation (48) can be easily deduced from the parasupersymmetric formulation. For this, we write the cyclic operators b and $b^\dagger$ in terms of the parafermionic operators a and $a^\dagger$ of order p as[2]

$$b \equiv a + \frac{a^{\dagger p}}{(p!)^2}, \qquad b^\dagger \equiv a^\dagger + \frac{a^p}{(p!)^2}. \tag{49}$$

Using the expression for the operators b and $b^\dagger$, we can readily check that the unity relation (48) simply is a direct consequence of equation (21). Apparently, for $p = 1$, the cyclic operators b and $b^\dagger$ turn into the Hermitian operator $f + f^\dagger$ given in equation (1).

[2] Even though the cyclic operators b and $b^\dagger$ are constructed directly from the parafermionic operators a and $a^\dagger$ by imposing cyclic symmetry, we will not call them the cyclic parafermionic operators. It is because that they do not obey the defining parafermionic algebra, the second and third equations of equation (6).

Furthermore, the cyclic operators b and $b^\dagger$ satisfy a general set of multilinear relations that resembles that of the parafermionic operators a and $a^\dagger$ (20). Explicitly, the set of multilinear relations for the cyclic operators of order p is given by

$$\sum_{m=0}^{p} b^{p-m} b^{\dagger\, p-q} b^m. \tag{50}$$

The computation of equation (50) is straightforward. It can be carried out by using matrix representation of the parafermionic operators a and $a^\dagger$, or alternatively, by using the method of mathematical deduction. We will only present the results. After some algebra, the general set of multilinear relations for the cyclic operators b and $b^\dagger$ can be established in the form

$$\sum_{m=0}^{p} b^{p-m} b^{\dagger\, p-q} b^m = \left[((p-q)!)^2 \binom{2p-q+1}{q} + \sum_{k=1}^{p-q} \frac{(p-k-q)!}{(p-k+1)!} \frac{(k-1)!}{(k+q)!}\right] b^q, \tag{51}$$

where on the right-hand side the first coefficient inside the square bracket is the same as that in equation (19), while the second one is a new contribution coming from the required cyclic symmetry. In particular, when setting $q = 0$ we obtain the expression, in which the right-hand side simplifies to a pure number,

$$\sum_{m=0}^{p} b^{p-m} b^{\dagger\, p} b^m = (p!)^2 + \sum_{k=1}^{p} \frac{1}{k(p-k+1)}. \tag{52}$$

Now, we construct the algebra associated with the cyclic charge operators in cyclic quantum mechanics of arbitrary order. In a similar fashion, the cyclic charge operators of order p, denoted by $\mathcal{Q}$ and $\mathcal{Q}^\dagger$, are given by the corresponding parasupercharges Q and $Q^\dagger$ of the same order as

$$\mathcal{Q} \equiv Q + \frac{Q^{\dagger\, p}}{\lambda^{p-1}}, \qquad \mathcal{Q}^\dagger \equiv Q^\dagger + \frac{Q^p}{\lambda^{p-1}}, \tag{53}$$

where the introduction of the dimensional parameter λ is to make the two cyclic charge operators $\mathcal{Q}$ and $\mathcal{Q}^\dagger$ having the correct mass dimension. This implies that λ has the dimension of mass. In fact, λ can be regarded as a tunable parameter that interpolates between cyclic quantum mechanics and PSQM of the same order. As can be expected that when $\lambda \to \infty$, the former theory is reduced to the latter one.

The algebra of cyclic quantum mechanics of order p is then characterized by the cyclic charge operators $\mathcal{Q}$ and $\mathcal{Q}^\dagger$ (53) that satisfy the following relations:

$$(\mathcal{Q})^{p+1} = (\mathcal{Q}^\dagger)^{p+1} = \frac{2^p}{\lambda^{p-1}} \prod_{i=1}^{p} (H - c_i), \qquad [H, \mathcal{Q}] = 0 = [H, \mathcal{Q}^\dagger], \tag{54}$$

and a set of multilinear relations (plus its Hermitian-conjugated set) shown in equation (61). Here, H is the cyclic Hamiltonian and has exactly the same form as the parasupersymmetric Hamiltonian given in equations (13) and (14). The proof of both relations in equation (54) is simple: the first relation is a direct consequence of equation (29) and the second relation is trivial, since H by definition commutes with Q and $Q^\dagger$.

With regard to the set of multilinear relations fulfilled by the cyclic charge operators of order p, it is given by the usual expression

$$\sum_{m=0}^{p} \mathcal{Q}^{p-m} \mathcal{Q}^{\dagger\, p-q} \mathcal{Q}^m. \tag{55}$$

In the same vein, this set of multilinear relations can be calculated for any arbitrary order. We report the final computational results. For the case of $p = 1$, there is only one such multilinear relation: $\mathcal{Q}\mathcal{Q}^\dagger + \mathcal{Q}^\dagger\mathcal{Q} = 4H$. For the case of $p = 2$, there are two multilinear relations that are

$$\mathcal{Q}^2\mathcal{Q}^\dagger + \mathcal{Q}\mathcal{Q}^\dagger\mathcal{Q} + \mathcal{Q}^\dagger\mathcal{Q}^2 = 2\left[\sum_{i=1}^{2}(H - c_i) + \frac{1}{\tilde{\lambda}^2}\prod_{i=1}^{2}(H - c_i)\right]\mathcal{Q}, \tag{56}$$

$$\mathcal{Q}^2\mathcal{Q}^{\dagger 2} + \mathcal{Q}\mathcal{Q}^{\dagger 2}\mathcal{Q} + \mathcal{Q}^{\dagger 2}\mathcal{Q}^2 = 2^2\prod_{i=1}^{2}(H - c_i)\left[1 + \frac{1}{\tilde{\lambda}^2}\sum_{i=1}^{2}(H - c_i)\right], \tag{57}$$

where for simplicity we have introduced the rescaled parameter $\tilde{\lambda} = \lambda/\sqrt{2}$. It is easily seen that, if the parameter $\tilde{\lambda} \to \infty$, equations (56) and (57) are simplified to equations (23) and (24), respectively. Furthermore, for the case of $p = 3$, we have three relations

$$\sum_{m=0}^{3}\mathcal{Q}^{3-m}\mathcal{Q}^\dagger\mathcal{Q}^m = 2\left[\sum_{i=1}^{3}(H - c_i) + \frac{1}{\tilde{\lambda}^4}\prod_{i=1}^{3}(H - c_i)\right]\mathcal{Q}^2, \tag{58}$$

$$\sum_{m=0}^{3}\mathcal{Q}^{3-m}\mathcal{Q}^{\dagger 2}\mathcal{Q}^m = 2^2\sum_{i=1}^{2}\left[\prod_{j=i}^{i+1}(H - c_j) + \frac{1}{\tilde{\lambda}^4}\frac{\prod_{j=1}^{3}(H - c_j)^2}{\prod_{k=i}^{i+1}(H - c_k)}\right]\mathcal{Q}, \tag{59}$$

$$\sum_{m=0}^{3}\mathcal{Q}^{3-m}\mathcal{Q}^{\dagger 3}\mathcal{Q}^m = 2^3\prod_{i=1}^{3}(H - c_i)\left[1 + \frac{1}{\tilde{\lambda}^4}\sum_{j=1}^{3}\frac{\prod_{k=1}^{3}(H - c_k)}{(H - c_j)}\right]. \tag{60}$$

The above three equations reduce to equations (25) to (27), as $\tilde{\lambda}$ goes to infinity.

Finally, for the case of general order p the set of multilinear relations satisfied by the cyclic charge operators of order p is found to be

$$\sum_{m=0}^{p}\mathcal{Q}^{p-m}\mathcal{Q}^{\dagger p-q}\mathcal{Q}^m = 2^{p-q}\left[\sum_{i=1}^{q+1}\mathcal{K}(i, p - q) + \frac{\mathcal{K}(1, p)^2}{\tilde{\lambda}^{2(p-1)}}\sum_{i=1}^{p-q}\frac{1}{\mathcal{K}(i, q + 1)}\right]\mathcal{Q}^q, \tag{61}$$

where the function $\mathcal{K}(i, n)$ is defined by

$$\mathcal{K}(i, n) \equiv \prod_{k=i}^{i+n-1}(H - c_k) = (H - c_i)(H - c_{i+1})\cdots(H - c_{i+n-1}). \tag{62}$$

There are n terms in the product function $\mathcal{K}(i, n)$. In particular, when $q = 0$, equation (61) takes the homogeneous bi-multilinear form in the cyclic charges $\mathcal{Q}$ and $\mathcal{Q}^\dagger$ as

$$\sum_{m=0}^{p}\mathcal{Q}^{p-m}\mathcal{Q}^{\dagger p}\mathcal{Q}^m = 2^p\mathcal{K}(1, p)\left[1 + \frac{\mathcal{K}(1, p)}{\tilde{\lambda}^{2(p-1)}}\sum_{i=1}^{p}\frac{1}{H - c_i}\right]. \tag{63}$$

We note that when taking $\tilde{\lambda} \to \infty$, equations (61) and (63) reduce to equations (28) and (29), respectively.

5. Conclusion

We have constructed, in the present paper, a general set of multilinear relations (28) that are fulfilled by the parasupercharges Q and $Q^\dagger$ in the framework of PSQM of arbitrary order. One immediate consequence regarding this set of multilinear relations is that a polynomial combination of the parasupersymmetric Hamiltonian H is shown expressible in terms of the associated parasupercharges (29). In particular, when all the energy parameters c_i, for $i = 1, 2, \ldots, p$, are set to zero, the parasupersymmetric Hamiltonian by itself is explicitly expressed by the corresponding parasupercharges (47). Moreover, based on our parasupersymmetric formulation, the translation of PSQM of order p to HSQM (nonlinear SQM) of the same order can be made quite transparent. Let us mention that the latter theory can also be obtained by the hidden supersymmetric structure in pure parabosonic systems, where the nonlinear supersymmetry is realized as a reduced parasupersymmetry [26].

We also study the structure of the energy spectrum of PSQM of order p in a systematic way. We divide the whole spectrum into the upper portion spectrum that is $(p+1)$-fold degenerate and the lower potion spectrum that is at most p-fold degenerate. Then, the degenerate structure of the lower portion spectrum can be analyzed by introducing the combined state defined in equation (42), which is a linear composition of the eigenfunctions (33) constructed from the corresponding superpotentials. We consequently establish the necessary condition for the lower portion spectrum to respect parasupersymmetry of order p. This condition is shown to be the parasupersymmetric generalization of that of the ground state for unbroken supersymmetry. Further, the reflection symmetry of the lower portion spectrum is discussed and the number of its distinct degenerate structures is determined.

In the final part of the paper, we introduce the notion of cyclic symmetry and develop the algebra associated with the cyclic charge operators $\mathcal{Q}$ and $\mathcal{Q}^\dagger$ in the cyclically symmetric quantum mechanics. The complete set of multilinear relations satisfied by the cyclic charge operators of arbitrary order can be constructed, based on the formulation of PSQM. Here, to define the cyclic charges consistently, a dimensional parameter λ needs to be introduced, which in turns describes the tunable parameter that interpolates between cyclic quantum mechanics and PSQM of the same order. Actually, the interpolating parameter λ has another interesting limit that is when $\lambda \to 0$. In this limit, the original set of multilinear relations (61) of the cyclic charges $\mathcal{Q}$ and $\mathcal{Q}^\dagger$ will transform into a brand-new set of multilinear relations of the parasupercharges Q and $Q^\dagger$. To illustrate this point, we take equation (63) as the example. In the limit $\tilde{\lambda} \to 0$, this equation becomes

$$\lim_{\tilde{\lambda}\to 0}\left[\tilde{\lambda}^{2(p-1)}\sum_{m=0}^{p}\mathcal{Q}^{p-m}\mathcal{Q}^{\dagger p}\mathcal{Q}^{m}\right] = 2^p\mathcal{K}(1,p)^2\sum_{i=1}^{p}\frac{1}{H-c_i}. \tag{64}$$

After the limiting procedure on the left-hand side is taken, the terms inside the square bracket can be shown to represent a more complicated multilinear combination of the parasupercharges Q and $Q^\dagger$. The detailed expression is omitted here.

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